数理解析研究所講究録 1290 巻 2002 年 7-24

On Hilbert's 17th Problem and Real Nullstellensatz for Definable Analytic Functions

京都大学・理学研究科・数学教室　藤田　雅人 (Masato Fujita)
Department of Mathematics, Graduate School of Science,
Kyoto University

名古屋大学・多元数理科学研究科　塩田　昌弘 (Masahiro Shiota)
Graduate School of Mathematics,
Nagoya University

1 Introduction.

David Hilbert proposed the 23 famous problems in his address at the 1900 International Congress of Mathematics in Paris. The following is the 17th prolem of them.

Hilbert's 17th Problem. Is every nonnegative polynomial function on $\mathbb{R}^n$ a sum of squares of rational functions?

Let A be a commutative ring. Remember that an ideal I of A is a real ideal if, for every sequence $a_1, \dots, a_p$ of elements of A with $a_1^2 + \cdots + a_p^2 \in I$, we have $a_1, \dots, a_p \in I$. On the other hand, we set

$$\mathcal{Z}_{\text{poly}}(I) := \{a \in \mathbb{R}^n; f(a) = 0 (\forall f \in I)\} \text{ and}$$
$$\mathcal{I}_{\text{poly}}(Z) := \{f \in R[x_1, \dots, x_n]; f(a) = 0 (\forall a \in Z)\}$$

for every ideal I of $\mathbb{R}[x_1, \dots, x_n]$ and for any subset Z of $\mathbb{R}^n$.

Real Nullstellensatz. Let I be an ideal of $\mathbb{R}[x_1, \dots, x_n]$. Then $\mathcal{I}_{\text{poly}}(\mathcal{Z}_{\text{poly}}(I)) = I$ if and only if I is a real ideal.

Hilbert's 17th Problem was solved by Emil Artin in 1927 [A] and it is known that above Real Nullstellensatz also holds true. Hilbert 17th Problem and Real Nullstellensatz are problems on polynomial functions. Real algebraic geometers solved the problems on Nash functions similiar to Hilbert's 17th Problem and Real Nullstellensats.

Hilbert's 17th Problem for Nash functions. Let R be a real closed field and $M \subset R^n$ be a semialgebraically connected Nash submanifold. Then every nonnegative Nash function on M is a sum of squares in the field of fractions of the ring $\mathcal{N}(M)$ of Nash functions on M.

Real Nullstellensatz for Nash functions. Let R be a real closed field and $M \subset R^n$ be a Nash submanifold. We set

$$\mathcal{Z}_{\text{Nash}}(I) := \{a \in M; f(a) = 0 (\forall f \in I)\} \text{ and}$$
$$\mathcal{I}_{\text{Nash}}(Z) := \{f \in \mathcal{N}(M); f(a) = 0 (\forall a \in Z)\}$$

for every ideal I of $\mathcal{N}(M)$ and for any subset Z of M. Then an ideal I of $\mathcal{N}(M)$ is a real ideal if and only if $\mathcal{I}_{\mathrm{Nash}}(\mathcal{Z}_{\mathrm{Nash}}(I)) = I$

An o-minimal expansion $\tilde{\mathbb{R}}$ of the real field is a collection of subsets of Euclidean spaces $\mathbb{R}^1$, $\mathbb{R}^2$, ... satisfying a few simple axioms which is satisfied for the collection of all semialgebraic subsets. In other word, the notion of o-minimal structures is a simple model-theoritical generalization of Real algebraic geometry. Therefore, it is natural to propose the problems whether the ring $C^\omega_{\mathrm{df}}(M)$ of definable analytic functions on M satisifies the following Hilbert's 17th Problem and Real Nullstellensatz for definable analytic functions for each o-minimal expansion $\tilde{\mathbb{R}}$ of the real field. Here M denotes a definable analytic submanifold of some Euclidean space.

Hilbert's 17th Problem for definable analyic functions. Assume that M is connected. Then every nonnegative definable analytic function on M is a sum of squares in the field of fractions of the ring $C^\omega_{\mathrm{df}}(M)$.

Real Nullstellensatz for definable analytic functions. We set

$$\mathcal{Z}(I) = \mathcal{Z}_{\mathrm{df}}(I) := \{a \in M; f(a) = 0 (\forall f \in I)\} \text{ and}$$
$$\mathcal{I}(Z) = \mathcal{I}_{\mathrm{df}}(Z) := \{f \in C^\omega_{\mathrm{df}}(M); f(a) = 0 (\forall a \in Z)\}$$

for every ideal I of $C^\omega_{\mathrm{df}}(M)$ and for any subset Z of M. Then an ideal I of $C^\omega_{\mathrm{df}}(M)$ is a real ideal if and only if $\mathcal{I}_{\mathrm{df}}(\mathcal{Z}_{\mathrm{df}}(I)) = I$.

We solve these problems in the low dimensional cases in the present paper. Precisely,

Theorem 1.1. *Consider an o-minimal expansion $\tilde{\mathbb{R}}$ of the real field and a definable analytic submanifold M of some Euclidean space. Assume further that one of the following conditions is satisfied.*

1. $\dim(M) \leq 1$
2. *All compact analytic subsets of Euclidean spaces are definable in $\tilde{\mathbb{R}}$ and* $\dim(M) = 2$
3. *The o-minimal structure $\tilde{\mathbb{R}}$ is the structure such that all compact analytic subsets of Euclidean spaces are definable, it admits an analytic cylindrical definable cell decomposition and the rings $C^\omega_{df}(N)$ are Noetherian for all definable analytic submanifold N of Euclidean spaces of dimension*$= 3$. *Assume further* $\dim(M) = 3$.

Then the following statements hold true.

Hilbert's 17th Problem. *Any nonnegative definable analytic function on M is a sum of squares in the field of fractions of the ring $C^\omega_{df}(M)$ when M is connected.*

Real Nullstellensatz. *An ideal I of $C^\omega_{df}(M)$ is a real ideal if and only if $\mathcal{I}_{df}(\mathcal{Z}_{df}(I)) = I$.*

2 Review: Real Algebra.

We review the notions concerning a real spectrum.

Definition 2.1 (prime cone). Let A be a unitary commutative ring. The subset α of A is a prime cone of A if α satisfies the following conditions.

1. $\alpha + \alpha \subset \alpha$
2. $\alpha \cdot \alpha \subset \alpha$
3. $a^2 \in \alpha$ for every a in A
4. $-1 \notin \alpha$
5. If $ab \in \alpha$ for $a, b \in A$, then $a \in \alpha$ or $-b \in \alpha$.

Notation 2.2. The set $\operatorname{supp}(\alpha) = \alpha \cap -\alpha$ is a real prime ideal of A. Let $k(\operatorname{supp}(\alpha))$ denote the field of fractions of $A/\operatorname{supp}(\alpha)$. The prime cone α induces an ordering $\leq_\alpha$ of the field $k(\operatorname{supp}(\alpha))$. The ordering is defined as follows.

$$0 \leq_\alpha \bar{a} \Longleftrightarrow a \in \alpha,$$

for every $a \in A$, where $\bar{a}$ denotes the class of a in $k(\operatorname{supp}(\alpha))$. We simply denote $a(\alpha) \geq 0$ if $\bar{a} \geq_\alpha 0$. We also define $a(\alpha) = 0$, $a(\alpha) < 0$ and $a(\alpha) > 0$ in the same way.

Definition 2.3 ($\operatorname{Spec}_r A$)**.** The real spectrum of A, denoted $\operatorname{Spec}_r A$, is the topological space whose points are the prime cones of A, and whose topology is given by the basis of open subsets

$$\mathcal{U}(a_1, \dots, a_n) = \{\alpha \in \operatorname{Spec}_r A; a_1(\alpha) > 0, \dots, a_n(\alpha) > 0\},$$

where $\{a_1, \dots, a_n\}$ is any finite family of elements of A. This topology is called the spectral topology.

Let α, β be two points of $\operatorname{Spec}_r A$. If the condition $\alpha \subset \beta$ is satisfied, we say that β is a specialization of α, or that α is generalization of β. We abbreviate this condition to $\alpha \to \beta$.

The following theorems are direct consequences of Real Algebra.

Theorem 2.4 (Hilbert's 17th Problem for a commutative field). *Let K be a commutative field. Define $\sum K^2$ as the set of all finite sums of squares of elemets of K. Then $\sum K^2 = \bigcap_{\alpha \in \operatorname{Spec}_r K} \alpha$.*

Proof. [BCR, Corollary 1.1.11] ■

Theorem 2.5 (Real Going-down for Regular Homomorphisms). *Let $\phi : A \to B$ be a homomorphism between commutative rings. Let $\beta \to \alpha$ in $\operatorname{Spec}_r(A)$, and let $\alpha' \in \operatorname{Spec}_r(B)$ be such that*

$$\phi^*(\alpha') := \{a \in A; \phi(a) \in \alpha'\} = \alpha.$$

Suppose that the local domain $A_{\beta\alpha} := A_{\mathrm{supp}(\alpha)}/\mathrm{supp}(\beta)$ *is excellent and the induced homomorphism* $A_{\beta\alpha} \to B \otimes A_{\beta\alpha}$ *is regular. Then there exists* $\beta' \in \mathrm{Spec}_r(B)$ *such that* $\beta' \to \alpha'$, $ht(\mathrm{supp}(\beta')) = ht(\mathrm{supp}(\beta))$ *and* $\phi^*(\beta') = \beta$.

Proof. [ABR, Theorem 7.1] ∎

Proposition 2.6 (Artin-Lang Property for convergent power series). *Let* $\mathbb{R}\{x_1, \dots, x_n\}$ *denote the ring of convergent power series. Let* α *be a prme cone of* $\mathbb{R}\{x_1, \dots, x_n\}$ *and* $f_1, \dots, f_p, g$ *be a sequence of elements of* $\mathbb{R}\{x_1, \dots, x_n\}$ *such that* $f_1(\alpha) > 0, \dots, f_p(\alpha) > 0, g(\alpha) = 0$. *We may regard* $f_1, \dots, f_p, g$ *as real analytic functions on an open neighborhood* U *at the origin in the* n*-dimensional Euclidean space. Then the germ of the set* $\{x \in U; f_1(x) > 0, \dots, f_p(x) > 0, g(x) = 0\}$ *at the origin is not empty.*

Proof. [R, Proposition 3.4] ∎

3 One-dimensional Case.

We prove Theorem 1.1 when $\dim(M) = 1$ in this section. We may assume that $M \subset \mathbb{R}^n$ is connected without loss of generality.

Let p be a prime ideal of $C^{\omega}_{\mathrm{df}}(M)$ with $\mathcal{Z}(p) \neq M$. Fix $x \in \mathcal{Z}(p)$. For all $a \in \mathbb{R}^n$, $S_{a,x}$ denote the polynomial functions on $\mathbb{R}^n$ defined by $S_{a,x}(y) = \mathrm{dist}(a, y)^2 - \mathrm{dist}(a, x)^2$. Here $\mathrm{dist}(a, x)$ denotes the distance between a and x. Choose a, b, c and $d \in \mathbb{R}^n$ suitably. Then we may assume the following.

- The zero set of $S_{a,x}$ intersects with M transversally and so does the zero set of $S_{b,x}$.
- $S^{-1}_{a,x}(0) \cap M = \{x, x'\}$ and $S^{-1}_{b,x}(0) \cap M = \{x, x''\}$
- $S^{-1}_{a,x}(0) \cap S^{-1}_{b,x}(0) \cap M = \{x\}$
- $S^{-1}_{c,x'}(0) \cap M = \{x'\}$ and $S^{-1}_{d,x''}(0) \cap M = \{x''\}$

From now on, $S_{a,x}$ also denotes the restiction of $S_{a,x}$ to M. This abuse of notation will not confuse the readers.

We first show that $p = m_x := \{f \in C^{\omega}_{\mathrm{df}}(M); f(x) = 0\}$ and p is finitely generated. We have only to show that p is generated only by $S_{a,x}$ and $S_{b,x}$. However, for any $f \in p$, $S_{c,x'}f$ is divisible by $S_{a,x}$ and $S_{d,x''}f$ is also divisible by $S_{b,x}$. Especially $S(a, x), S(b, x) \in p$ because p is prime. Remark that $\pm S_{c,x'} \pm S_{d,x''}$ is an unit in $C^{\omega}_{\mathrm{df}}(M)$ if we choose the signs properly. Therefore, $f \in (S_{a,x}, S_{b,x})$.

Proof of Real Nullstellesatz. It is obvious when I is a prime ideal because all prime ideals are of the form m_x or (0). It is also trivial that I is real if $\mathcal{I}(\mathcal{Z}(I)) = I$

Next consider the case when I is a real ideal. Remark that a real ideal is a radical ideal. Consider the irreducible primary decomposition $I = p_1 \cap \cdots \cap p_m$.

Since I is radical, we may assume that all p_j are prime. We show that all p_j are real. Let $f_1, \dots, f_r$ be definable analytic functions with $f_1^2 + \dots + f_r^2 \in p_j$. Choose $h \in \bigcap_{i \neq j} p_i \setminus p_j$. Then $(hf_1)^2 + \dots + (hf_r)^2 \in I$. Since I is real, all hf_k are in I. Particularly, hf_k is in p_j. Therefore, $f_k \in p_j$ because $h \notin p_j$. We have shown that all p_j are real. Since $p_j = I_{\mathrm{df}}(\mathcal{Z}(p_j))$ for all j, it is obvious that $I = I_{\mathrm{df}}(\mathcal{Z}(I))$. ■

Proof of Hlbert's 17th Problem. Any nonnegative definable analytic function f is obviously divisible by $S_{a,x}^2 + S_{b,x}^2$ if $f(x) = 0$. Hence $f = u \prod_{i=1}^m (S_{a_i,x_i}^2 + S_{b_i,x_i}^2)^{m_i}$, where u is a positive function on M. We have finished the proof. ■

Remark 3.1. For any ring A and elements $a, b, c, d \in A$, the equation $(a^2 + b^2)(c^2 + d^2) = (ad - bc)^2 + (ac + bd)^2$ holds true. Hence the above proof of Hilbert's 17th Problem claims the stronger fact that any nonnegative definable analytic function on M is a sum of two definable analytic functions on M if $\dim(M) = 1$.

4 Basic lemmas

We first fix an arbitrary o-minimal expansion $\tilde{\mathbb{R}}$ of the real field and a definable submanifold M of an Euclidean space.

Lemma 4.1. *Any maximal ideal m of $C^\omega_{df}(M)$ is of the form*

$$m_x = m_x^M := \{f \in C^\omega_{df}(M); f(x) = 0\},$$

where $x \in M$.

Proof. Any real analytic set is locally homeomorphic to an union of finite cones at any point by [S, Theorem II, p.96]. Remark that, if a real analytic set is locally homeomorphic to some Euclidean space at some point, the germ of this analytic set at the given point is analytically irreducible as a germ of real analytic set. Since an o-minimal structure admits finite definable cell decompositions, any definable real analytic set particularly has only finite analytically irreducible components. Whence, for any family Ψ of definable analytic functions on M, there exists finite definable analytic functions $f_1, \dots, f_k \in \Psi$ with $\bigcap_{f \in \Psi} f^{-1}(0) = \bigcap_{i=1}^k f_i^{-1}(0)$.

There particularly exist finite elements $f_1, \dots, f_k \in m$ such that $\bigcap_{f \in m} \mathcal{Z}(f) = \bigcap_{i \leq i \leq k} \mathcal{Z}(f_i)$. If the zero set of $\sum_{i=1}^k f_i^2$ is empty, $\sum_{i=1}^k f_i^2$ is invertible. Contradiction. Choose a point x of the zero set of the function $\sum_{i=1}^k f_i^2$. It is obvious that $m \subset m_x^M$ because m is maximal. ■

Lemma 4.2. *Let $\alpha \in Spec_r(C^\omega_{df}(M))$ be a prime cone. Then the set*

$$\{x \in M; f_1(x) \geq 0, \dots, f_m(x) \geq 0\}$$

is not empty for any finite family $f_1, \dots, f_m \in \alpha$.

Proof. We show it by the reduction to absurdity. As was shown in the proof of Lemma 4.1, there exist finite elements $h_1, \dots, h_k \in \mathrm{supp}(\alpha)$ such that $\bigcap_{h \in \mathrm{supp}(\alpha)} h^{-1}(0) = \bigcap_{1 \le i \le k} h_i^{-1}(0)$. We set $f_{m+1} = -\sum_{i=1}^k h_i$.

Set $F_j = \{x \in M; f_j(x) \ge 0\}$ and $G_j = \{x \in M; f_i(x) \ge f_j(x) \text{ for any } i = 1, \dots m+1\}$ for all $j = 1, \dots m+1$. The intersection $F_j \bigcap G_j$ is empty because $(\{x \in M; f_i(x) < 0\})_{i=1,\dots,m+1}$ is an open covering of M by the definition. Define the definable continuous function $\phi : M \to \mathbb{R}$ as follows.

$$\phi(x) = (m+2) \times \frac{|\max_{i=1,\dots,m+1} f_i(x)|}{|\min_{i=1,\dots,m+1} f_i(x)|}$$

We will construct positive definable analytic functions $P_1, \dots, P_{m+1}$ on M satisfying the following conditions for any j.

$$P_j > \phi \text{ on } G_j$$
$$P_j < 1 \text{ on } F_j$$

Fix the number j. Arranging (f_j) in a suitable order, we may assume that $j = m+1$. First set

$N = \mathbb{R}^{m+1} \setminus \{(x_1, \dots, x_m, y) \in \mathbb{R}^{m+1}; y \ge 0, x_i \ge 0 \text{ for all } i\}$,
$F = \{(x_1, \dots, x_m, y) \in N; y \ge 0\}$ and
$G = \{(x_1, \dots, x_m, y) \in N; x_i \ge 0 \text{ for all } i\}$.

Let d_F and d_G denote the distance function from F and G, respectively. We next define two kinds of semialgebraic functions Q and ψ_r on N, where r is a positive number.

$$Q(x_1, \dots, x_m, y) = \sqrt{y^2 + (x_1 + y)^2 + \cdots + (x_m + y)^2} \times (m+2)$$
$$\psi_r(x_1, \dots, x_m, y) = \frac{r d_G}{d_F + d_G} Q - \frac{y d_F}{2(d_F + d_G)}$$

It is easy to see that $\psi_r > 0$ on N and $\psi_{r'} \ge \psi_r$ on N if $r' > r$. There exists a Nash function P' on N with $|\psi_{\frac{3}{2}} - P'| < \psi_{\frac{1}{2}}$ by Efroymson's Approximation Theorem. Obviously, $P' > 0$ on N, $P' < |y|$ on G and $P' > Q$ on F. Hence, $Q/P' > (m+2) \times \dfrac{|\max\{x_1 + y, \dots, x_m + y, y\}|}{|\min\{x_1 + y, \dots, x_m + y, y\}|}$ on G and $Q/P' < 1$ on F. The function P_{m+1} on M defined by

$$P_{m+1}(x) = \frac{Q(f_1(x) - f_{m+1}(x), \dots, f_m(x) - f_{m+1}(x), f_{m+1}(x))}{P'(f_1(x) - f_{m+1}(x), \dots, f_m(x) - f_{m+1}(x), f_{m+1}(x))}$$

satisfies the required condition.

Any positive definable analytic function g on M is contained in α because $g = (\sqrt{g})^2$ and $\sqrt{g}$ is a definable analytic function. Particularly, $P_j \in \alpha$. Hence the definable analytic function $\sum_{i=1}^{m+1} P_i f_i$ is contained in α; nevertheless, it is negative on M because $|P_j f_j| > (m+2)|\max f_j|$ when f_j reaches to the minimum value among $(f_i)_{i=1,\dots,m+1}$. Therefore it is not contained in α. Contradiction. ■

We next consider an o-minimal expansion $\tilde{\mathbb{R}}$ of the ristricted analytic field $\mathbb{R}_{\mathrm{an}}$. Remember that the o-minimal structure $\mathbb{R}_{\mathrm{an}}$ is by definition the smallest o-minimal expansion of the real field such that all compact analytic subsets of Euclidean spaces are definable. Hence all compact analytic subsets of Euclidean spaces are definable in any o-miniaml expansion $\tilde{\mathbb{R}}$ of $\mathbb{R}_{\mathrm{an}}$.

Proposition 4.3. *Consider an o-minimal expansion $\tilde{\mathbb{R}}$ of $\mathbb{R}_{an}$. Let M be a definable analytic submanifold of $\mathbb{R}^n$ and N be a definable open subset of M. Assume that $C^\omega_{df}(M)$ and $C^\omega_{df}(N)$ are both noetherian. Then the localization $C^\omega_{df}(M)_p$ is an excellent and regular local ring for any prime ideal p. Furthermore, the natural ring homomorphisms $C^\omega_{df}(M) \to C^\omega_{df}(N)$ and $C^\omega_{df}(M) \to \mathcal{O}_x$ are regular homomorphisms, where x is a point of M.*

Proof. It is obvious that the Krull dimension of $C^\omega_{\mathrm{df}}(M)_{m_x}$ is more than or equal to $\dim(M)$ for $x \in M$. We first show that $C^\omega_{\mathrm{df}}(M)_{m^M_x}$ is a regular ring whose Krull dimension $\dim(M)$ and that $m^N_x C^\omega_{\mathrm{df}}(N)_{m^N_x}$ is generated by m^M_x.

Fix a point $x \in M$. We may assume without loss of generality that $x = (0,\dots,0,1)$ and that M is contained in the unit sphere S^{n-1}. Choose a linear function f_1 on $\mathbb{R}^n$ such that $x \in f_1^{-1}(0)$, $f_1^{-1}(0)$ is transversal to M at x and the connected component of $f_1^{-1}(0) \cap M$ containing x is compact. Let $C^\omega(\bar{M})$ be the ring of germs of analytic functions on $\bar{M}$. Since any connected component of an analytic set is also an analytic set, there exists an analytic function on $\tilde{h} \in C^\omega(\bar{M})$ with $\tilde{h}(x) \neq 0$ such that $\tilde{h}f$ is divisible by f_1 for any analytic function f on M vanishing on X.

Let m_X be the ideal of analytic functions on X vanishing at x. Since X is a coherent analytic set, one can easily show that m_X is generated by the coordinate functions $x_1,\dots,x_n$ by Cartan's Theorem B [Cartan, Théorème 3]. It is obvious that $\dim(M)-1$ of them, say, $x_1,\dots,x_{\dim(M)-1}$ is nonsingular at x on X and the other coordinate functions are generated by $x_1,\dots,x_{\dim(M)-1}$ as elements of $C^\omega(X)_{m_X}$. Particularly, $C^\omega(X)_{m_X}$ is a regular local ring and there exist $f'_2,\dots,f'_{\dim(M)} \in C^\omega(X)$ which generates the maximal ideal m_X in $C^\omega(X)_{m_X}$. By Cartan's Theorem B, there exist analytic functions $f_2,\dots,f_{\dim(M)}$ on S^{n-1} such that the restriction of f_j to a neighborhood of X coincides with f'_j for any $j = 2,\dots,\dim(M)$.

We show that $m^M_x C^\omega_{\mathrm{df}}(M)_{m^M_x}$ is generated by $f_1,\dots f_{\dim(M)}$. Fix a definable analytic function f on M vanishing at x. There exist analytic functions $g'_2,\dots,g'_{\dim(M)}$ and h' on X such that $h'f = \sum_{i=2}^{\dim(M)} g'_i f_i$ on X and $h'(x) \neq 0$. Choose an analytic extension $g_2,\dots,g_{\dim(M)}$ and h of $g'_2,\dots,g'_{\dim(M)}$ and h' on S^{n-1}, respectively. It is possible by Cartan's Theorem B. By the way of construction, $hf - \sum_{i=2}^{\dim(M)} g_i f_i$ is an analytic function on M vanishing on X. Therefore there exists an analytic function g_1 on M with $g_1 f_1 = \tilde{h}hf - \sum_{i=2}^{\dim(M)} \tilde{h}g_i f_i$, that is, $\tilde{h}hf = g_1 f_1 + \sum_{i=2}^{\dim(M)} \tilde{h}g_i f_i$. All f_i, g_j, h and $\tilde{h}$ are definable for $i = 1,\dots,\dim(M)$ and for $j = 2,\dots,\dim(M)$ because they are analytic on S^{n-1} or on $\bar{M}$. It is obvious that g_1 is definable. We have shown that $C^\omega_{\mathrm{df}}(M)_{m^M_x}$ is a regular local ring whose Krull dimension is $\dim(M)$. The ring $C^\omega_{\mathrm{df}}(M)$ is regular and its Krull dimension equals $\dim(M)$ by Lemma 4.1 and by

[M1, Theorem 19.3]. We can show in the same way that $C^\omega_{\mathrm{df}}(M)_{m_x^M}$ is excellent using [M2, Theorem 102]. We omit the details. Remark that $f_1,\dots,f_{\dim(M)}$ also generate the ideal $C^\omega_{\mathrm{df}}(N)_{m_x^N}$.

We next show that the natural ring homomorphisms $C^\omega_{\mathrm{df}}(M)\to C^\omega_{\mathrm{df}}(N)$ and $C^\omega_{\mathrm{df}}(M)\to \mathcal{O}_x$ are regular homomorphisms. The proof of that $C^\omega_{\mathrm{df}}(M)\to\mathcal{O}_x$ is regular is similar to that of the regularity of the homomorphism $C^\omega_{\mathrm{df}}(M)\to C^\omega_{\mathrm{df}}(N)$. Therefore we only show the regularity of the homomorphism $\phi: C^\omega_{\mathrm{df}}(M)\to C^\omega_{\mathrm{df}}(N)$. Fix $x\in N$. Consider the following diagram.

$$\begin{tikzcd} C^\omega_{\mathrm{df}}(M)_{m_x^M} \arrow[r] \arrow[d] & \widehat{C^\omega_{\mathrm{df}}(M)_{m_x^M}} \arrow[d] \\ C^\omega_{\mathrm{df}}(N)_{m_x^N} \arrow[r] & \widehat{C^\omega_{\mathrm{df}}(N)_{m_x^N}} \end{tikzcd}$$

The second vertical arrow is an isomorphism because $\phi(m_x^M)=m_x^N$. Since both $C^\omega_{\mathrm{df}}(M)_{m_x^M}$ and $C^\omega_{\mathrm{df}}(N)_{m_x^N}$ are both excellent rings, the first horizontal arrows are regular and faithfully flat. The first vertical arrow is therefore a regular homomorphism by [M1, Theorem 32.1]. Summing up, the homomorphism ϕ is regular by Lemma 4.1. ■

Corollary 4.4. *Let $\tilde{\mathbb{R}}$ and M be the same as in Proposition 4.3. If the first cohomology class $H^1(M,\mathbb{Z}_2)$ is zero, then $C^\omega_{df}(M)$ is an unique factorization domain.*

Proof. The proof is exactly the same as [BE, Theorem 4.1]. Hence we omit the proof. ■

5 Two-dimensional Case.

We fix an o-minimal expansion $\tilde{\mathbb{R}}$ of $\mathbb{R}_{\mathrm{an}}$ in this section.

Fix a prime cone α of $C^\omega_{\mathrm{df}}(M)$. We set Γ_α as the family of all closed definable subsets of the form

$$S=\{y\in M; f_1(y)\geq 0,\dots,f_m(y)\geq 0\},$$

where $f_1,\dots,f_m$ is a finite family of elements in α. Remark that any element of Γ_α is not empty by Lemma 4.2.

Lemma 5.1. *Fix a prime cone α of $C^\omega_{df}(M)$. Assume that the intersection $\bigcap_{S\in\Gamma_\alpha} S$ is not an empty set. Then the set $\bigcap_{S\in\Gamma_\alpha} S$ consists of only one point x and the prime cone has the specialization α_x, where $\alpha_x=\{f\in C^\omega_{df}(M); f(x)\geq 0\}$.*

Proof. Obvious. ■

Theorem 5.2. *Consider an o-minimal expansion $\tilde{\mathbb{R}}$ of $\mathbb{R}_{an}$ and a 2-dimensional definable analytic submanifold M of some Euclidean space. Then $C^\omega_{df}(M)$ is a Noetherian ring.*

Proof. Let M be contained and bounded in $\mathbb{R}^n$ and we can also assume that M is connected. Let $\tilde{M}^{\mathbb{C}}$ and $M^{\mathbb{C}}$ denote a complexification of M and its germ at M, respectively. (A complexification is always considered in $\mathbb{C}^n$ and of class C^ω.) Here note that $M^{\mathbb{C}}$ is unique but $\tilde{M}^{\mathbb{C}}$ is not so. We set $\mathcal{O}^{\mathbb{C}} = \mathcal{O} \otimes_{\mathbb{R}} \mathbb{C}$ and define $\tilde{\mathcal{O}}^{\mathbb{C}}$ to be the sheaf of holomorphic functions on $\tilde{M}^{\mathbb{C}}$.

For a subset p of $C^\omega(M)$, let $\mathcal{Z}(p)$ denote the common zero set of p and be called a global analytic set. We next define $\mathcal{Z}^{\mathbb{C}}(p)$ the germ at M of a complex analytic set of some $\tilde{M}^{\mathbb{C}}$ as follows. The sheaf $p\mathcal{O}$ of $\mathcal{O}$-ideals generated by p is coherent. It is easy to choose $\tilde{M}^{\mathbb{C}}$ where $p\mathcal{O}^{\mathbb{C}}$ is extendable as a coherent sheaf of $\tilde{\mathcal{O}}^{\mathbb{C}}$-ideals. Let $\widetilde{p\mathcal{O}^{\mathbb{C}}}$ denote such an extension. We define $\mathcal{Z}^{\mathbb{C}}(p)$ to be the germ at M of $\operatorname{supp}(\tilde{\mathcal{O}}^{\mathbb{C}}/\widetilde{p\mathcal{O}^{\mathbb{C}}})$. We regard $\mathcal{Z}^{\mathbb{C}}(p) \subset M^{\mathbb{C}}$ and $M \subset M^{\mathbb{C}}$. Remark $\mathcal{Z}(p) = \operatorname{supp}(\mathcal{O}/p\mathcal{O})$ and $\mathcal{Z}(p) = \mathcal{Z}^{\mathbb{C}}(p) \cap M$. The notation $\dim_{\mathbb{C}} \mathcal{Z}(p)$ denotes the complex dimension of $\mathcal{Z}^{\mathbb{C}}(p)$. We call a germ of the form $\mathcal{Z}^{\mathbb{C}}(p)$ a complex analytic set germ (at M). It is known that a complex analytic set admits a unique decomposition into irreducible ones.

We first show that a prime ideal p of $C^\omega_{\mathrm{df}}(M)$ is finitely generated if $\mathcal{Z}(p)$ is compact. Remark that $pC^\omega(M)$ is finitely generated. Hence there exists a finite family $f_1, \dots, f_m$ of elements of p which generates $pC^\omega(M)$. Chaging $f_1, \dots, f_m$ with $f_1 + K(\sum_{j=1}^m f_j^2), \dots, f_m + K(\sum_{j=1}^m f_j^2)$ for some large K, we may assume that $f_i^{-1}(0)$ are compact for all i. Fix an arbitrary element f of p. Remark that $(f_1, \dots, f_m)C^\omega(M) = H^0(M, (f_1, \dots, f_m)\mathcal{O})$ by Cartan's Theorem B. Since $\mathcal{Z}(p)$ is compact, $f^l \in (f_1, \dots, f_m)C^\omega(M)$ by Hilbert Nullstellensatz. Let $\bar{\mathcal{I}}$ mean the trivial extension of a sheaf $\mathcal{I}$ of $\mathcal{O}_M$-ideals with compact support to one of $\mathcal{O}_{\mathbb{R}^n}$-ideals defined by $\bar{\mathcal{I}}_x = \mathcal{O}_x$ for $x \in \mathbb{R}^n \setminus M$. By Cartan's theorem B, the sequences

$$0 \to H^0(M, f_1\mathcal{O}) \to H^0(M, \mathcal{I}(A)) \xrightarrow{q} H^0(M, \mathcal{I}(A)/f_1\mathcal{O}) \to 0$$
$$H^0(\mathbb{R}^n, \overline{(f_2, \dots, f_k)\mathcal{O}}) \xrightarrow{q_{\mathbb{R}^n}} H^0(\mathbb{R}^n, \mathcal{I}(A)/\overline{f_1\mathcal{O}})$$

are exact, and we can identify $H^0(M, \mathcal{I}(A)/f_1\mathcal{O})$ with $H^0(\mathbb{R}^n, \mathcal{I}_{\mathbb{R}^n}(A)/\overline{f_1\mathcal{O}})$. Hence there is $h_1 \in H^0(\mathbb{R}^n, \overline{(f_2, \dots, f_k)\mathcal{O}})$ such that $q_{\mathbb{R}^n}(h_1) = q(f)$, and, therefore, $f - h_1|_M \in H^0(M, f_1\mathcal{O})$. Let $g_1 \in C^\omega(M)$ be such that $f - h_1|_M = g_1 f_1$. Then $h_1|_M$ and, hence, g_1 are definable. In the same way, we can construct $h_2 \in H^0(\mathbb{R}^n, \overline{(f_3, \dots, f_k)\mathcal{O}})$ and $g_2 \in C^\omega_{\mathrm{df}}(M)$ such that $h_1|_M - h_2|_M = g_2 f_2$. Repeating these arguments, we can construct $g_3, \dots, g_k$ satisfying the required property.

We next show that a prime ideal p of $C^\omega_{\mathrm{df}}(M)$ is finitely generated if $\mathcal{Z}(p)$ is not compact. We want to show that $\mathcal{Z}^{\mathbb{C}}(p)$ has no analytically irreducible cmponents C such that $C \cap M$ is compact. Assume the contrary.

If $\dim_{\mathbb{C}}(C) = 1$, there exists a definable analytic function h on M such that $\mathcal{Z}^{\mathbb{C}}(h) = C$ and any analytic function vanishing on C is divisible by h in $C^\omega(M)$ as shown before. Remark that $h \notin p$ by the definition of C. Fix an arbitrary element f of p. Then $f/h^k \in p$ vanishes on all analytically irreducible components of $\mathcal{Z}^{\mathbb{C}}(p)$ except C for some $k \in \mathbb{N}$. Contradiction to the definition

If $\dim_{\mathbb{C}}(C) = 0$, we can easily construct a definable analytic function $f \in p$ such that any analytically irreducible component Y of $\mathcal{Z}^{\mathbb{C}}(f)$ containing C satisfies the condition $\dim(Y \cap M) = 0$. Therefore, as in the case when $\dim_{\mathbb{C}}(C) = 1$, we can construct a definable analytic function $g \in p$ such that $\mathcal{Z}^{\mathbb{C}}(g)$ has no analytically irreducible components containing C except $\{Y_i\}$, where $\{Y_i\}$ denote the analytically irreedubible components of $\mathcal{Z}(p)$ containing C. Contradiction to the definition of C. We have shown that $\mathcal{Z}^{\mathbb{C}}(p)$ has no analytically irreducible cmponents C such that $C \cap M$ is compact.

Let $\{X_i\}$ be the irreducible components of $\mathcal{Z}^{\mathbb{C}}(p)$. Set $X = \bigcup_i X_i$. We want to see $p = I_{\text{df}}(X)$. Clearly $p \subset I_{\text{df}}(X)$. Assume the contrary. Remark that $I_{\text{df}}(X)\mathcal{O}_x$ is principal for each $x \in M$. Obviously $I_{\text{df}}(X)^2$ is also principal by [Cain]. Let f be its generator and choose $0 \neq g \in p$. Then g^2/f^m for some $m \in \mathbb{N}$ is an element of p, but does not vanish on some X_i. This is a contradiction.

We show that p is finitely generated. Choose an analytically irreducible component X_1. Remark that $I_{\text{df}}(X_1) = p$. There exists a $f \in I_{\text{df}}(X)$ whose multiplicity on X_1 is just 1. It is easy to construct $g \in p$ such that $(g^{\mathbb{C}})^{-1}(0) = X$ and $g \geq 0$ on M. Let $\{Y_j\}$ be analytically irreducible components of $(f^{\mathbb{C}})^{-1}(0)$ each one of which is different from all X_i. Choose one point $y_j \in M \cap Y_j$. Then we can choose $c_j \in \mathbb{R}$ such that $g_j(y_j) \neq 0$ and the multiplicity of g_j on X_1 is 1, where $g_j = f + c_j g$. Let $\{Z_k\}$ be analytically irreducible components of $(f^{\mathbb{C}})^{-1}(0) \cap \bigcap_j (g_j^{\mathbb{C}})^{-1}(0)$. Choosing points and real numbers, we can construct a finite family of definable analytic functions whose complex common zero set is strictly smaller than $\cup Z_k$ and which have the multiplicity 1 on X_1. Continuing in this way, we can find $f_1, \dots, f_l \in I_{\text{df}}(X_1)$ with multiplicity 1 on X_1 and $\bigcap_{t=1}^{l} (f_t^{\mathbb{C}})^{-1}(0)$ coincides with X.

Fix an arbitrary $h \in I_{\text{df}}(X_1)$. Remark that $I_{\text{df}}(Y)^2$ is principal for all analytically irreducible component Y of $M^{\mathbb{C}}$ of complex dimension= 1. Hence, we can construct definable analytic functions h_t such that hh_t is divisible by f_t and $\bigcap_{t=1}^{l} (h_t^{\mathbb{C}})^{-1}(0) = \emptyset$. Therefore $h = \sum_{t=1}^{l} \dfrac{h_t^2 h/f_t}{\sum_{t'=1}^{l} h_{t'}^2} f_t$. We have shown that $I_{\text{df}}(X_1)$ is generated by f_t. ■

Theorem 5.3. *Consider an o-minimal expansion $\tilde{\mathbb{R}}$ of $\mathbb{R}_{an}$. Let M be a 2-dimensional definable analytic manifold, and let α be a prime cone of $C^{\omega}_{df}(M)$ with supp(α) $= (0)$. Assume that $f_1(\alpha) > 0, \dots, f_s(\alpha) > 0$ for given $f_1, \dots, f_s, g \in C^{\omega}_{df}(M)$. Then $f_1(y) > 0, \dots, f_s(y) > 0$ for some $y \in M$.*

Proof. Remark that $C^{\omega}_{\text{df}}(N)$ is noetherian for any open set N of M. We first consider the case when $\bigcap_{S \in \Gamma_\alpha} S$ is not empty. The prime cone α is a generalization of $\alpha_x := \{f \in C^{\omega}_{\text{df}}(M); f(x) \geq 0\}$ by Lemma 5.1. Let $\phi : C^{\omega}_{\text{df}}(M) \to \mathcal{O}_x$ be a canonical homomorphism. Remark that $\phi^*(\alpha_{\textbf{ana},x}) = \alpha_x$, where $\alpha_{\textbf{ana},x}$ denotes a prime cone $\{f \in \mathcal{O}_x; f(x) \geq 0\}$. Therefore, there exists a prime cone β of $\mathcal{O}_x$ such that $\phi^*(\beta) = \alpha$ by Theorem 2.5. Hence $f_1(\beta) > 0, \dots, f_p(\beta) > 0, g(\beta) = 0$. The set $\{x \in M; f_1(x) > 0, \dots, f_p(x) > 0, g(x) = 0\}$ is not empty by Proposition

We next consider the other case. Set $d(\alpha) = \min\{\dim(S); S \in \Gamma_\alpha\}$. If $d(\alpha) = 2$, this theorem is trivial. It is also trivial in the case when g is not a zero function because the set

$$\{y \in M; f_1(y) \geq 0, \dots, f_s(y) \geq 0, g(y) = 0\}$$

is not enpty by Lemma 4.2. Therefore we only consider the case when $\bigcap_{S\in\Gamma_\alpha} S = \emptyset$, $d(\alpha) = 1$ and $g \equiv 0$ on M.

We lead the contradiction with the assumption that the set

$$\{y \in M; f_1(y) > 0, \dots, f_s(y) > 0\}$$

is empty. We first prove a claim which is necessary for the proof of this theorem. We denote $m = \mathrm{mult}_Y(f)$ for any definable analytic function f on M if and only if $f \in p_Y^m$ and $f \notin p_Y^{m+1}$, where Y is a 1-dimensional analytically irreducible subset of M and p_Y is the ideal of germs of analytic functions on M at a nonsingular point $y \in Y$ which vanish on Y. It is well known that $\mathrm{mult}_Y(f)$ is independent of the choice out of the nonsingular point y. Remark that there exists a natural extension $\mathrm{mult} : C^\omega_{\mathrm{df}}(M)_{p_Y} \to \mathbb{N} \cup \{0\}$.

Claim. Let Y denote a 1-dimensional definable analytic subset of M such that $p_Y = I_{\mathrm{df}}(Y)$ is a prime ideal. Then $p_Y C^\omega_{\mathrm{df}}(M)_{p_Y}$ is a prime ideal generated by a definable function h_Y with $\mathrm{mult}_Y(h_Y) = 1$.

Proof of Claim. The ring $C^\omega_{\mathrm{df}}(M)_{p_Y}$ is a regular local ring. It is especially an unique factorization Noetherian domain by [M1, Theorem 20.3]. Hence, p_Y is a principal ideal by [M1, Theorem 20.1]. Since the natural mapping $T\mathbb{R}^n \supset \{(y, v) \in M \times T_y\mathbb{R}^n\} \to TM$ is definable and analytic, there exists an $\mathbb{R}$-derivation $D : C^\omega_{\mathrm{df}}(M)_{p_Y} \to C^\omega_{\mathrm{df}}(M)_{p_Y}$ such that $\mathrm{mult}_Y(D(f)) = \mathrm{mult}_Y(f) - 1$ for any $f \in C^\omega_{\mathrm{df}}(M)_{p_Y}$ with $\mathrm{mult}_Y(f) > 0$. In fact, we have only to take the derivation induced by the image of $\sum_{k=1}^n a_k \dfrac{\partial}{\partial x_k}$, where a_k are real numbers, such that it is not zero in T_yM at some nonsingular point $y \in X$. Therefore the generator h_Y of $p_Y C^\omega_{\mathrm{df}}(M)_{p_Y}$ has multiplicity $\mathrm{mult}_Y(h_Y) = 1$. Claim is proved.

The analytic closure of the set $Z(F) = \{y \in M; g_1(y) \geq 0, \dots, g_p(y) \geq 0\}$ in M is a 1-dimensional analytic set for any finite family $F = \{g_1, \dots, g_p\} \subset \alpha$ with $\dim Z(F) \neq 2$. This analytic closure has only finite analytically irreducible components as was shown in the proof of Lemma 4.1. At least one analytically irreducible analytic set Y is an analytically irreducible component of the analytic closure of $Z(F)$ for all finite families $F \subset \alpha$ with $\dim Z(F) \neq 2$ because any element of Γ_α has dimension ≥ 1 by the assumption.

Fix a family $F = \{g_1, \dots, g_p\}$ with $\dim Z(F) = 1$. Assume that there exists $S \in \Gamma_\alpha$ such that, for any $y \in S \cap Z(F) \cap Y$, the relations $g_i|_Y \equiv 0$ and $g_i|_{U_y} \leq 0$ hold true for some $i = 1, \dots, p$, where U_y is a sufficiently small open neighborhood of y in M. There exist $w_i, z_i \in C^\omega_{\mathrm{df}}(M) \setminus I_{\mathrm{df}}(Y)$ such that $w_i g_i = z_i h_Y^{\mathrm{mult}_Y(g_i)}$ by Claim. We may assume that $w_i, z_i, h_Y \in \alpha$ without loss of generality. Obviously, $w_i \cdot z_i$ is not positive on U_y because $\text{mult}_Y(g_i)$ is even. Furthermore, $Y \not\subset w_i^{-1}(0) \cup z_i^{-1}(0)$ by the definition of w_i and z_i. Therefore the analytic closure of $Z(F') \in \Gamma_\alpha$ does not have Y as an analytically irreducible component, where $F' = F \cup \{w_i \cdot z_i; i = 1, \ldots, p\}$. Contradiction to the definition of Y.

Consider the case when $F = \{f_1, \ldots, f_s\}$. By Claim, $v_i f_i = u_i h_Y^{\text{mult}_Y(f_i)}$ for some $v_i, u_i \in \alpha \cap (C^\omega_{\text{df}}(M) \setminus I_{\text{df}}(Y))$. Set $F' = \{f_i, u_i, v_i; i = 1, \ldots, s\}$. As was shown in the last paragraph, there exists a point $y \in Y \cap Z(F')$ such that, for any open neighborhood U of y in M, $U \cap \phi^{-1}((0,\infty)) \neq \emptyset$ and y is not contained in any 1-dimensional analytically irreducible component of $\phi^{-1}(0)$ except Y for any $\phi \in F'$. Fix such a small neighborhood U of y. The equation $f_q|_Y = f_r|_Y \equiv 0$ holds true and $f_q \cdot f_r$ is negative on $U \setminus Y$ for some q and r. We may assume that $\text{mult}_Y(f_q) \geq \text{mult}_Y(f_r)$ without loss of generality. Remark that u_q, u_r, v_q and v_r are positive on U. These facts, however, contradicts to the equation $u_r v_q f_q = u_q v_r f_r h_Y^{\text{mult}_Y(f_q) - \text{mult}_Y(f_r)}$ because the number $\text{mult}_Y(f_q) - \text{mult}_Y(f_r)$ is an even number. ■

Proof of Theorem 1.1 when $\dim(M) = 2$. We first show Hilbert's 17th Problem. Let K be the quotient field of $C^\omega_{\text{df}}(M)$. Assume that $f \notin \alpha$ for some positive cone of K. Since $-f(\alpha) > 0$, $-f(x) > 0$ for some $x \in M$ by Theorem 5.3. Contradiction. Therefore, $f \in \bigcap_{\alpha \in \text{Spec}_r(K)} \alpha$. Theorem is proved by Theorem 2.4.

We next show Real Nullstellensatz. We have only to show as in the case when $\dim(M) = 1$ that $\mathcal{I}(\mathcal{Z}(p)) = p$ if p is a real prime ideal. Real Nullstellensatz is obvious when $\dim(\mathcal{Z}(p)) = 1$ because $\dim(C^\omega_{\text{df}}(M)) = 2$. If $\dim(\mathcal{Z}(p)) = 0$, choose a point $x \in \mathcal{Z}(p)$. Consider the homomorphism $C^\omega_{\text{df}}(M) \to \mathcal{O}_x$. Remark that there exists a prime cone with support p if p is a real prime ideal. Hence, there exists a prime cone α' of $\mathcal{O}_x$ such that $\text{ht}(\text{supp}(\alpha')) = 1$ and $\dim(\mathcal{Z}(\text{supp}(\alpha))) = 0$ by Theorem 2.5. Contradiction to Proposition 2.6. ■

6 Three-dimensional Case.

In this section, $\tilde{\mathbb{R}}$ denotes an o-minimal expansion of $\mathbb{R}_{\text{an}}$ such that it admits an analytic cylindrical definable cell decompositon and the rings $C^\omega_{\text{df}}(N)$ are Noetherian rings for all 3-dimensional definable analytic submanifold N of Euclidean spaces. Remark that $C^\omega_{\text{df}}(N)$ are Noetherian rings for all 3-dimensional definable analytic submanifold N of Euclidean spaces when $\tilde{\mathbb{R}} = \mathbb{R}_{\text{an}}$ ([FS]). The purpose of this section is to finish the proof of Theorem 1.1.

Lemma 6.1. *Let M be a bounded definable analytic submanifold of $\mathbb{R}^n$. Let α be a prime cone of $C^\omega_{df}(M)$ such that $C \in \Gamma_\alpha$ for some 1-dimensional definable set C. Then one of the following prime cones is a specialization of α.*

Points of C $\alpha_x = \{f \in C^\omega_{df}(M); f(x) \geq 0\}$, *where $x \in C$.*

Curve germ *The prime cone $\alpha_{C,x}$ defined as follows, where x is a point in*

$\bar{C} \setminus C$. *The function* f *is an element of* $\alpha_{C,x}$ *if and only if the closure of the set* $C \cap \{y \in M; f(y) \geq 0\}$ *in* $\mathbb{R}^n$ *contains the point* x.

Proof. When the intersection $\bigcap_{S \in \Gamma_\alpha} S$ contains a point x, the prime cone α obviously has a specialization α_x. We consider the case when this intersection is empty. Since the closure $\bar{M}$ of M is compact, the set $T = \bigcap_{S \in \Gamma_\alpha} \bar{S}$ is not empty and contained in $\bar{C} \setminus C$. The set T consists of only one point x because any two points are separated by some definable analytic function. It is easy to see by using Lemma 4.2 that the closure of $C \cap \{y \in M; f(y) \geq 0\}$ contains the point x for any $f \in \alpha$. ∎

Lemma 6.2. *Let* $\alpha \in Spec_r(C^\omega_{df}(\mathbb{R}))$. *Suppose that* $f_1(\alpha) > 0, \dots, f_s(\alpha) > 0, g(\alpha) = 0$ *for some* $f_1, \dots, f_s, g \in C^\omega_{df}(\mathbb{R})$. *Then* $f_1(x) > 0, \dots, f_s(x) > 0, g(x) = 0$ *for some* $x \in \mathbb{R}$.

Proof. Remark that $\operatorname{supp}(\alpha)$ is a maximal ideal or (0). When $\operatorname{supp}(\alpha)$ is a maximal ideal, $\alpha = \alpha_x$ for some $x \in \mathbb{R}$. Therefore, $f_1(x) > 0, \dots, f_s(x) > 0, g(x) = 0$.

We consider the case when $\operatorname{supp}(\alpha) = (0)$ next. Remark that $g \equiv 0$ in this case. One can easily show that any definable analytic function f is factorized as follows.

$$f = u \prod_{i=1}^n (t - a_i),$$

where all a_i are real numbers and u is an unit. We may therefore assume that the analytic functions f_i are of the form $f_i = t - a_i$. If there exist no points $x \in \mathbb{R}$ such that $f_1(x) > 0, \dots, f_s(x) > 0$, there exists $a \in \mathbb{R}$ such that both $t - a$ and $a - t$ are in α by Lemma 4.2. Contradiction to the assumption that $\operatorname{supp}(\alpha) = (0)$. ∎

Let f_1 and f_2 be (not necessarily analytic) definable functions on $(0, \infty)$. We denote $f_1 \sim f_2$ if $f_1|_{(0,r)} \equiv f_2|_{(0,r)}$ for some $r > 0$. The relation $\sim$ is then an equivalence relation. We set $K(\tilde{\mathbb{R}})$ as the set of equivalence classes with respect to this relation.

Lemma 6.3. *The set* $K(\tilde{\mathbb{R}})$ *is a real closed field and the prime cone* $\alpha_{(0,\infty),0}$ *of* $C^\omega_{df}((0,\infty))$ *defined in Lemma 6.1 lies over the prime cone* $K(\tilde{\mathbb{R}})^2$ *under the natural injection* $C^\omega_{df}((0,\infty)) \hookrightarrow K(\tilde{\mathbb{R}})$.

Proof. It is obvious that $K(\tilde{\mathbb{R}})$ is a field and $K(\tilde{\mathbb{R}})^2 = \{[f]; f|_{(0,r)} \geq 0 \text{ for some } r\}$, where $[f]$ represents the equivalence class of a definable function f. Clearly, $K(\tilde{\mathbb{R}})^2$ is a positive cone. See [ABR] for the definition of a positive cone.

Fix a polynomial $P(t, X) \in K(\tilde{\mathbb{R}})[X]$ of odd degree. There exists a positive number r such that $P(t, X) = \sum_{k=0}^n f_k(t) X^k$, where $f_k(t)$ is a definable function on $(0, r)$ with $f_n(t) \neq 0$ for any $t \in (0, r)$. Since $\mathbb{R}$ is a real closed field, there exists a root of $P(t, X)$ in $\mathbb{R}$ for any $t \in (0, r)$. Hence there exists a definable function $g : (0, r') \to \mathbb{R}$ with $P(t, g(t)) \equiv 0$. Namely, $P(t, X)$ has a root in

$K(\tilde{\mathbb{R}})$. Therefore, $K(\tilde{\mathbb{R}})$ is a real closed field by [BCR, Theorem 1.2.2]. The last part of this lemma is obvious by the definition. ■

Theorem 6.4. *Let M be a 3-dimensional definable analytic submanifold of $\mathbb{R}^n$ and let α be a prime cone of $C^\omega_{df}(M)$ with support (0). Assume that $f_1(\alpha) > 0, \ldots, f_s(\alpha) > 0$ for given $f_1, \ldots, f_s \in C^\omega_{df}(M)$. Then $f_1(y) > 0, \ldots, f_s(y) > 0$ for some $y \in M$.*

Proof. We may assume that M is connected and bounded in $\mathbb{R}^n$. The proof of this theorem is the same as that of Theore 5.3 in the case when $\bigcap_{S\in\Gamma_\alpha} \neq \emptyset$. Hence we may assume that the set $\bigcap_{S\in\Gamma_\alpha}$ is empty. Set $d(\alpha) = \min\{\dim(S); S \in \Gamma_\alpha\}$. By the definition of $d(\alpha)$, $d(\alpha) \neq 0$. This theorem is trivial in the case when $d(\alpha) = 3$. We can prove this theorem in the same way as Theorem 5.3 when $d(\alpha) = 2$. Therefore we may also assume that α has a specialization $\alpha_{C,x}$. Here C is a 1-dimensional definable analytic submanifold of M, x is a point in the boundary of C in $\mathbb{R}^n$, and $\alpha_{C,x}$ is a prime cone of $C^\omega_{\mathrm{df}}(M)$ defined in Lemma 6.1.

Take a definable analytic tubler neighborhood N of C in M and the definable analytic retraction $\rho : N \to C$. Let $\alpha'_{C,x}$ be the prime cone of $C^\omega_{\mathrm{df}}(N)$ such that $h \in \alpha'_{C,x}$ if and only if the closure of the set $C \cap \{y \in N; h(y) \geq 0\}$ contains the point x. Then there exists a prime cone α' of $C^\omega_{\mathrm{df}}(N)$ which is lying over α by Lemma 6.1, Proposition 4.3 and Theorem 2.5. Take a shorter curve C and a smaller tubler neighborhood N. Then we can easily construct a definable and analytic diffeomorphism $(N, C) \simeq ((0, \infty) \times \mathbb{R}^{n-1}, (0, \infty) \times \{0\})$. Hence we may assume $M = (0, \infty) \times \mathbb{R}^{n-1}$, $C = (0, \infty) \times \{0\}$ and $x = (0, \ldots, 0)$. Let t be the first coordinate function and $x_1, \ldots, x_{n-1}$ be the other coordinate functions. We may further assume that all x_j belong to α and that $x_j = f_j$ for $j = 1, \ldots, n-1$ without loss of generality. We will construct functions $h_1, \ldots, h_s \in C^\omega_{\mathrm{df}}((0, \infty))[x_1, \ldots, x_{n-1}]$ such that the set $\{y \in U; f_1(y) > 0, \ldots, f_s(y) > 0\}$ contains the set $\{y \in U; h_1(y) > 0, \ldots, h_s(y) > 0\}$, where U is some open neighborhood of $(0, r) \times \{0\} \subset C$ and r is a positive real number. We first finish the proof with the assumption that such functions h_j are constructed in advance.

Set $\rho(t)$ as the maximal positive number such that $U \cap (\{t\} \times \mathbb{R}^{n-1})$ contains the cube $\{0\} \times (-\rho(t), \rho(t))^{n-1}$ for any $0 < t < r$. Since the function $\rho : (0, r) \to \mathbb{R}$ is definable, the restriction $\rho|_{(0,r')}$ is analytic. Set $D = \{(t, x_1, \ldots, x_{n-1}) \in (0, r') \times \mathbb{R}^{n-1}; -\rho(t) < x_i < \rho(t) \text{ for any } i.\}$. Then there exists a prime cone $\tilde{\alpha} \in \mathrm{Spec}_r(C^\omega_{\mathrm{df}}(D))$ lying over α' by Theorem 2.5, Lemma 6.1 and Proposition 4.3. Consider the embedding $\iota : D' = (0, r') \times (-1, 1)^{n-1} \to M$ given by $\iota(t, x_1, \ldots, x_{n-1}) = (t, \rho(t)x_1, \ldots \rho(t)x_{n-1})$. This embedding obviously induces a definable and analytic diffeomorphism between D' and D. The functions $h_j \circ \iota$ on D' are also in $C^\omega_{\mathrm{df}}((0, r'))[x_1, \ldots, x_n]$. The integral domain $C^\omega_{\mathrm{df}}((0, r'))$ is contained in the real closed field $K(\tilde{\mathbb{R}})$ by Lemma 6.3. The canonical homomorphism $C^\omega_{\mathrm{df}}((0, r'))[x_1, \ldots, x_{n-1}] \to K(\tilde{\mathbb{R}})[x_1, \ldots, x_{n-1}]$ is regular and faithfully flat by [M2, 33.B Lemma 1] and [M1, Theorem 7.2]. We show that $C^\omega_{\mathrm{df}}((0, r'))[x_1, \ldots, x_{n-1}]$ is regular and excellent. Regularity follows from

[M1, Theorem 19.5] because $C^\omega_{\mathrm{df}}((0,r'))$ is regular. Excellentness follows from [M2, Theorem 73, 77]. Therefore some prime cone α_{poly} of $K(\tilde{\mathbb{R}})[x_1,\dots,x_{n-1}]$ lies over $\tilde{\alpha}$ because the prime cone β_{poly} consisting of all polynomials whose constant terms are positive lies over $\alpha_{C,x}$. By Positivstellensatz, there exists a tuple $(x_1(t),\dots,x_{n-1}(t)) \in K(\tilde{\mathbb{R}})$ such that $h_1(x_1(t),\dots,x_{n-1}(t)) > 0,\dots,h_s(x_1(t),\dots,x_{n-1}(t)) > 0$ as elements of $K(\tilde{\mathbb{R}})$. The definable curve γ given by $\gamma(t) = (t, x_1(t),\dots,x_{n-1}(t))$ is well defined for any small enough positive real number t. Particularly, the set $\{y \in U; f_1(y) > 0,\dots,f_s(y) > 0\}$ is not empty.

We give some notations. Fix a definable analytic function f. Set $c_{(i_1,\dots,i_{n-1})} = c^f_{(i_1,\dots,i_{n-1})} = \dfrac{1}{i_1!\cdots i_{n-1}!}\dfrac{\partial^{i_1+\cdots+i_{n-1}} f}{\partial x_1^{i_1}\cdots\partial x_{n-1}^{i_{n-1}}}(t,0,\dots,0) \in C^\omega_{\mathrm{df}}((0,\infty))$. For $k = 1,\dots,n-1$, define the natural number $\kappa^k_1 = \kappa^k_1(f)$ satisfying that f is not in the ideal generated by $x_k^{\kappa^k_1(f)+1}$ but in that generated by $x_k^{\kappa^k_1(f)}$. We also define the number $\kappa^k_2 = \kappa^k_2(f)$ satisfying that $c^f_{(\kappa^1_1,\dots,\kappa^{k-1}_1,l,\kappa^{k+1}_1,\dots,\kappa^{n-1}_1)} \equiv 0$ for any $l < \kappa^k_2$ and $c_{(\kappa^1_1,\dots,\kappa^{k-1}_1,\kappa^k_2,\kappa^{k+1}_1,\dots,\kappa^{n-1}_1)} \not\equiv 0$. Remark that, for any definable analytic function g on M, there exist definable analytic functions $c(t), d_1(t,x_1), d_2(t,x_1,x_2),\dots,d_{n-1}(t,x_1,\dots,x_{n-1})$ on M with $g = c + \sum_{i=1}^{n-1} d_i x_i$ because $M = (0,\infty)\times\mathbb{R}$. In fact, we have only to set

$$d_j = \begin{cases} \dfrac{g(t,x_1,\dots,x_j,0,\dots,0) - c(t) - \sum_{i=1}^{j-1} x_i d_i(t,x_1,\dots,x_i)}{x_j} & \text{if } x_j \neq 0 \\[2ex] \dfrac{\partial(g - c - \sum_{i=1}^{j-1} x_i d_i)}{\partial x_j}(t,x_1,\dots,x_{j-1},0,\dots,0) & \text{otherwise} \end{cases}$$

which is analytic because the numerator vanishes on $\{x_i = 0\}$. Set

$$\begin{aligned} I(f) &:= \{i = (i_1,\dots,i_{n-1}); 0 \le i_1 \le \kappa^1_2(f),\dots,0 \le i_{n-1} \le \kappa^{n-1}_2(f)\} \text{ and} \\ J(f) &:= \{j = (j_1,\dots,j_{n-1}); 0 \le j_1 \le \kappa^1_2(f)+1,\dots,0 \le j_{n-1} \le \kappa^{n-1}_2(f)+1, \\ &\qquad \exists k, j_k = \kappa^k_2(f)+1\}. \end{aligned}$$

Then there exist $d_{(j_1,\dots,j_{n-1})} = d^f_{(j_1,\dots,j_{n-1})} \in C^\omega_{\mathrm{df}}(M)$ with

$$f_q = \textstyle\sum_{i\in I(f)} c_i x_1^{i_1}\cdots x_{n-1}^{i_{n-1}} + \sum_{j\in J(f)} d_j x_1^{j_1}\cdots x_{n-1}^{j_{n-1}}.$$

The prime cone $\alpha \cap C^\omega_{\mathrm{df}}((0,\infty))$ coincides with the prime cone $\alpha_{(0,\infty),(0)}$ defined in Lemma 6.1 by Lemma 6.2. Therefore, for any nonzero $c \in C^\omega_{\mathrm{df}}((0,\infty))$ and $P_j \in C^\omega_{\mathrm{df}}(M)$, the value $c(\alpha) + \sum_{j=1}^{n-1} x_j(\alpha)P_j(\alpha)$ at α is positive if and only if $c(\alpha)$ is positive by Lemma 4.2. Fix a definable analytic function f on M with $f(\alpha) > 0$.

Claim. $P(\alpha)x_k(\alpha)^{\kappa^k_2(f)+1}$ is smaller than $f(\alpha)$ for any $P \in C^\omega_{\mathrm{df}}(M)$ and for any k.

Proof of Claim. Assume that $P(\alpha)x_1(\alpha)^{\kappa^1_2(f)+1}$ is larger than $f(\alpha)$ for some

$P \in C^{\omega}_{\rm df}(M)$. Consider the case when $\kappa^1_1(f) \neq \kappa^1_2(f)$. We construct a definable analytic function satisfying $\kappa^1_2(g) = \kappa^1_2(f)$, $\kappa^1_1(g) > \kappa^1_1(f)$ and that $P(\alpha)x_1(\alpha)^{\kappa^1_2(g)+1}$ is larger than $g(\alpha)$ for some $P \in C^{\omega}_{\rm df}(M)$. Set $Q = f$ if $c^f_{\kappa^1_2(f),\kappa^2_1(f),\dots,\kappa^{n-1}_1(f)}(\alpha) < 0$ and set $Q = Px_1^{\kappa^1_2(f)+1} - f$ otherwise. By the definition, $Q(\alpha) > 0$. Furthermore, there exist definable analytic functions

$$P_0(t, x_2, \dots, x_{n-1}), \dots, P_{\kappa^1_2(f)-1}(t, x_2, \dots, x_{n-1}),\ R(t, x_1, \dots, x_{n-1})$$

on M such that $Q = \sum_{i=0}^{\kappa^1_2(f)-1} P_i x_1^i + x_1^{\kappa^1_2(f)} R$ and $R(\alpha) < 0$. Then $g = Q - x_1^{\kappa^1_2(f)} R$ satisfies the required property. Therefore we have only to prove this claim in the case when $\kappa^1_1(f) = \kappa^1_2(f)$. When $\kappa^1_1(f) = \kappa^1_2(f)$, the equation $\kappa^k_1(f) = \kappa^k_2(f)$ holds for any k by the definition.
Set $Q = f$ if $c^f_{\kappa^1_1(f),\kappa^2_1(f),\dots,\kappa^{n-1}_1(f)}(\alpha) < 0$ and set $Px_1^{\kappa^1_2(f)+1} - f$ otherwise. By the definition, $Q(\alpha) > 0$. The function Q, however, is of the form

$$Q = \prod_{i=1}^{n-1} x_i^{\kappa^i_1(f)}(c(t) + \textstyle\sum_{j=1}^{n-1} x_j P'_j),$$

where $c(t)$ is a definable analytic function on $(0, \infty)$ with $c(\alpha) < 0$ and P'_j are definable analytic functions on M. Particularly, $Q(\alpha) < 0$. Contradiction. We have shown the claim.

We construct $h_1, \dots h_s$ from $f_1, \dots, f_s$. Choose an arbitrary f_q. When $1 \leq q \leq n-1$, set $h_q = f_q = x_q$. We construct h_q for $q \geq n$. Any $d^{f_q}_{(j_1,\dots,j_{n-1})}$ is of the form $C^q_{(j_1,\dots,j_{n-1})} + \sum_{i=1}^{n-1} x_i P^q_{(j_1,\dots,j_{n-1}),i}$, where $C^q_{(j_1,\dots,j_{n-1})}$ is a definable analytic function on $(0, \infty)$ and $P^q_{(j_1,\dots,j_{n-1}),i}$ is a definable analytic function on M. Set

$$T^q_{(j_1,\dots,j_{n-1})} = -1 - \textstyle\sum_{i=1}^{n-1} x_i P^q_{(j_1,\dots,j_{n-1}),i}.$$

Then $T^q_{(j_1,\dots,j_{n-1})}(\alpha) < 0$. Set

$$\begin{aligned} J_q &= \{(j_1, \dots, j_{n-1}); 0 \leq j_1 \leq \kappa^1_2(f_q) + 1, \dots, 0 \leq j_{n-1} \leq \kappa^{n-1}_2(f_q) + 1, \\ &\qquad \exists k, j_k = \kappa^k_2(f_q) + 1\}, \\ h_q &= f_q + \sum_{(j_1,\dots,j_{n-1}) \in J} T^q_{(j_1,\dots,j_{n-1})} x_1^{j_1} \cdots x_{n-1}^{j_{n-1}} \text{ and} \\ U &= \bigcap_{q=n}^{s} \bigcap_{(j_1,\dots,j_{n-1}) \in J_q} \{y \in M; T^q_{(j_1,\dots,j_{n-1})}(y) < 0\}. \end{aligned}$$

Then $\bigcap_{q=1}^{s} h_q^{-1}((0,\infty)) \cap U \subset \bigcap_{q=1}^{s} f_q^{-1}((0,\infty)) \cap U$ by the way of construction. Since $T^q_{(j_1,\dots,j_{n-1})}$ is negative on C, U is an open neighborhood of C. As was shown before, $0 < -\sum_{(j_1,\dots,j_{n-1}) \in J} T^q_{(j_1,\dots,j_{n-1})}(\alpha) x_1(\alpha)^{j_1} \cdots x_{n-1}(\alpha)^{j_{n-1}} < f_q(\alpha)$ for $n \leq q \leq s$. Hence the constructed h_q and U satisfy the required properties. ■

Proof of Theorem 1.1 in the case when $\dim(M) = 3$. The proof of Hilbert's 17th Problem is the same as in the case when $\dim(M) = 2$. Hence we omit the proof of Hilbert's 17th Problem.

We will show Real Nullstellensatz. As shown in the proof of Real Nullstellensatz in lower dimensional case, we have only to prove that $\mathcal{I}_{\mathrm{df}}(\mathcal{Z}_{\mathrm{df}}(p)) = p$ when p is a real prime ideal. We can prove Real Nullstellensatz in the same way as in the lower dimensional case if $\mathcal{Z}_{\mathrm{df}}(p)$ is compact or height of p is 2. Hence we have only to show that p is not a real ideal when $\dim \mathcal{Z}_{\mathrm{df}}(p) = 1$ and $\mathrm{ht}(p) = 1$.

Assume the contrary. There exists a prime cone α of $C^{\omega}_{\mathrm{df}}(M)$ whose support coincides with p. If the intersection $\bigcap_{S \in \Gamma_\alpha} S$ is not empty, α has a specialization α_x by Lemma 5.1. There exists a prime cone β of $\mathcal{O}_x$ lying over α with $\mathrm{ht}(\mathrm{supp}(\beta)) = 1$ by Proposition 4.3 and Theorem 2.5. Since the germ of zeros of $\mathrm{supp}(\beta)$ is of dimension ≤ 1, it contradicts to Proposition 2.6. We next consider the case when $\bigcap_{S \in \Gamma_\alpha} S = \emptyset$. The prime cone α has a specialization $\alpha_{C,x}$ such that C is a definable analytic submanifold of M contained in $\mathcal{Z}(p)$ by Lemma 6.1. Choose a sufficiently short curve C, then we may assume that there exists a tubler neighborhood N of C such that the pair (N, C) is definably and analytically diffeomorphic to the pair $((0,\infty) \times \mathbb{R}^2, (0,\infty) \times \{0\})$. As was mentioned some times in the present paper, there exists a prime cone β of $C^{\omega}_{\mathrm{df}}(N)$ lying over α with $\mathrm{ht}(\mathrm{supp}(\beta)) = \mathrm{ht}(p) = 1$ by Proposition 4.3 and Theorem 2.5. It is obvious that $\mathcal{Z}(\mathrm{supp}(\beta))$ is of dimension $= 1$. Set $q = \mathrm{supp}(\beta)$, then q is generated by one definable analytic function f on N by Corollary 4.4 and [M1, Theorem 20.1]. We show that f and $-f$ are not a sum of finite squares of elements of the quotient field of $C^{\omega}_{\mathrm{df}}(N)$. Assume the contrary, then there exist definable analytic functions $P_1, \dots, P_m, Q \not\equiv 0$ on N with $Q^2 f = P_1^2 + \cdots + P_m^2$. Since q is real, all P_j are in q. Hence, $Q^2 = (P_1'^2 + \cdots + P_m'^2) f$ for some definable analytic functions $P_1', \dots, , P_m'$ because $C^{\omega}_{\mathrm{df}}(N)$ is a domain. There therefore exists a definable analytic function Q' on N with $Q = fQ'$ and $Q'^2 f = P_1'^2 + \cdots + P_m'^2$. Continuing in this way, we obtain that $Q \in \bigcap_{n \in \mathbb{N}} q^n$. The function Q vanishes on N by Krull intersection theorem [M1, Theorem 8.10]. Contradiction. Whence, f changes the sign on some open set which intersects with $f^{-1}(0)$ by Hilbert's 17th Problem. However, the zero set $f^{-1}(0)$ is of codimension > 1. They contradicts each other. We have shown Real Nullstellensatz in the case when I is a prime ideal. ■

References

[ABR] C. Andradas, L. Bröcker and J. -M. Ruiz, Constructible sets in real geometry, Springer, 1996.

[A] E. Artin, Über die Zerlegung definiter Funktionen in Quadrate, Hamb. Abh. 5(1927) 100-115, The collected papers of Emil Artin, Reading:Addison-Wwsley (1965) 273-288.

[BCR] J. Bochnak, M. Coste, M. -F. Roy. Real Algebraic Geometry, Springer, 1998.

[BE] J. Bochnak, G. A. Efroymson, Real algebraic geometry and the 17th Hilbert problem, Math. Ann. 251(1980) 213-241.

[Cain] B. E. Cain, A two-color theorem for analytic maps in $\mathbb{R}^n$, Proc. of Amer. Math. Soc. 39(2)(1973) 261-266.

[Cartan] H. Cartan, Variétés analytiques réelles et variétes analytique complexes, Bull. Soc. Math. France, 85(1957) 77-99.

[FS] M. Fujita and M. Shiota, Rings of analytic functions definable in o-minimal structure, 2002, preprint.

[M1] H. Mataumura, Commutative Ring Theory, Cambridge Univ. Press, 1986.

[M2] H. Matsumura, Commutative Algebra 2nd edition, Benjamin Reading, 1980.

[R] J. M. Ruiz, The basic theory of power series, Friedr. Vieweg & Sohn, 1993.

[S] M. Shiota, Geometry of subanalytic and semialgebraic sets, Birkhäuser,